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On stokeslets in a two-fluid space

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SUMMARY

A uniform force is applied over an arbitrarily orientated bounded plane area in the interior of a semi-infinite incompressible viscous fluid overlain by a dissimilar fluid. Based upon the Papkovitch–Neuber approach to the displacement equations of equilibrium in the theory of elasticity, it is shown that for any orientation of the force and loaded area, the velocities and stresses in the two phases can be found very simply by applying the *same* set of differential operators on the corresponding flow fields for a single homogeneous fluid occupying the whole space. A specialization of this theorem admits interpretations in terms of plate bending and extension.

1. Introduction

Solutions of problems involving application of internal concentrated forces are of fundamental interest in continuum mechanics. Apart from being Green's functions, they can be used, as demonstrated in the great papers of Eshelby [1–4], in the construction of solutions of more complicated and physically more realizable problems. The problems of dislocation, disclination, phase transformations and internal motions in sediment laden streams are cases in point.

The fundamental solution for a point force in an incompressible homogeneous infinite viscous fluid is available in the text-book of Lamb [5], while the case of a semi-infinite fluid satisfying the usual no-slip boundary condition has been thoroughly treated by Blake [6] who used integral transform techniques. Reference may also be made to two additional interesting papers by Blake and Chwang [7] and Blake [8] on semi-infinite viscous fluids.

The purpose of this paper is to consider the problem of a uniform force applied over a finite plane area of arbitrary orientation in the interior of one of two mutually immiscible semi-infinite incompressible viscous fluids. Such a two-layered system represents the simplest case of a stratified flow that can be used in studying the interaction of singularities with fluid-fluid interfaces. Our method of solution is that of taking advantage of the analogy between hydrodynamics and elastostatics. By making use of Papkovitch–Neuber potentials [9], suitably constructed to account for the orientation of the loaded area, we avoid the integrations of Blake [6] and obtain expressions in terms of elementary functions.

More importantly, a simple theorem is offered which shows that if we know the flow fields when a region in the hydrodynamic whole space is subjected to any force distribution, the corresponding results for the two-fluid space can be obtained by differentiation of those for the whole space. Thus the two-fluid problem under the present investigation belongs to the class of hydrodynamic problems discussed by Milne–Thomson [10], Butler [11], and Weiss [12], as well as the elasticity problems discussed by the author [13–15].

It is believed that the theorem offered here is the first step towards the analysis of velocity distributions in contiguous fluid layers of infinite lateral extent. From the excellent papers of Hetenyi [16–17] on elastic quarter planes and spaces under surface loads, it is also believed that for given singularities in the interior of a quarter-space incompressible viscous fluid, the prescribed no-slip conditions at the intersecting planes can be satisfied by repeated application of the theorem offered here. Mathematically, we first solve the hydrodynamic half-space problem by determining the Papkovitch potentials satisfying the conditions on the first plane. The resultant potentials will, however, violate the conditions to be satisfied on the second plane. To neutralize the velocities at this plane, we must superpose additional potentials with the aid of the theorem. This will, in turn, generate residual velocities on the first plane, which will have to be readjusted with the aid of the theorem to zero, thereby disturbing the no-slip condition on the second plane, and so on. Of course, the series solution obtained through such a process of repeated application must be tested for convergence.

2. The basic equations

When the inertia and body forces are negligible everywhere, the governing Navier–Stokes equations of steady motion for an incompressible viscous fluid are

$$\nabla P = \mu\nabla^2 \boldsymbol{U}, \ \nabla\cdot\boldsymbol{U} = 0, \tag{1}$$

$$\boldsymbol{\sigma} = -P\boldsymbol{I} + \mu(\nabla\boldsymbol{U} + \boldsymbol{U}\nabla). \tag{2}$$

In equations (1–2), $\boldsymbol{U}$, $\boldsymbol{\sigma}$ and P are the fluid velocity vector, stress tensor, and pressure, respectively, while $\boldsymbol{I}$ denotes the idemfactor. The constant μ designates the coefficient of viscosity, while the symbols ∇ and ∇^2 are the gradient and Laplacian operators.

As pointed out by the author in [18], a general solution of the vector equations (1), analogous to the Papkovitch–Neuber solution of the equations governing linearly elastic isotropic incompressible materials, admits the representation

$$\boldsymbol{U} = \nabla(\Psi_0 + \boldsymbol{r}\cdot\boldsymbol{\Psi}) - 2\boldsymbol{\Psi}, \ P = 2\mu\nabla\cdot\boldsymbol{\Psi} \tag{3}$$

provided that

$$\nabla^2\Psi_0 = \nabla^2\boldsymbol{\Psi} = 0 \tag{4}$$

and $\boldsymbol{r}$ denotes the three-dimensional position vector. Apart from completeness [19] and the wealth of knowledge about harmonic functions, the representation (3) allows us to keep in close contact with linear elastostatics. It may be noted that a general solution of (1) can also be taken in terms of the Galerkin [9] vector function in the form

$$\boldsymbol{U} = \nabla(\nabla\cdot\boldsymbol{\chi}) - \nabla^2\boldsymbol{\chi}, \ P = \mu\nabla^2(\nabla\cdot\boldsymbol{\chi}) \tag{5}$$

with

$$\nabla^2\nabla^2\boldsymbol{\chi} = 0. \tag{6}$$

3. The theorem

Taking x, y, and z as the usual rectangular Cartesian coordinates, our main result may be summarized as follows:

Theorem. Let $\boldsymbol{U}$ *and* P *be the velocity and pressure distributions for an arbitrary steady motion of an incompressible homogeneous infinite viscous fluid such that* $\boldsymbol{U}$ *and* P *have all their singularities in the lower half-space* $z > 0$ *(region 1). Suppose now that the upper half-space* $z < 0$ *(region 2) is filled with a different incompressible viscous fluid, leaving the distribution of singularities of* $\boldsymbol{U}$ *and* P *unchanged. Then, provided that the conditions*

$$\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{n} = \boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{n},\ \ \boldsymbol{U}^{(1)} = \boldsymbol{U}^{(2)} \tag{7}$$

are satisfied when $z = 0$, *the new flow fields for the two phases are given by, for* $z > 0$,

$$\begin{aligned} U_x^{(1)} &= U_x - A\bar{U}_x + Az\left[2\frac{\partial}{\partial x}\bar{U}_z + z\nabla^2\bar{U}_x\right],\\ U_y^{(1)} &= U_y - A\bar{U}_y + Az\left[2\frac{\partial}{\partial y}\bar{U}_z + z\nabla^2\bar{U}_y\right],\\ U_z^{(1)} &= U_z - A\bar{U}_z + Az\left[2\frac{\partial}{\partial z}\bar{U}_z - z\nabla^2\bar{U}_z\right],\\ P^{(1)} &= P - A\bar{P} + 2A\frac{\partial}{\partial z}[2\mu^{(1)}\bar{U}_z + z\bar{P}]; \end{aligned} \tag{8}$$

and for $z < 0$,

$$\boldsymbol{U}^{(2)} = (1 - A)\boldsymbol{U},\ \ P^{(2)} = (1 + A)P, \tag{9}$$

where

$$A = (\Gamma - 1)/(\Gamma + 1),\ \ \Gamma = \mu^{(2)}/\mu^{(1)},\ \ -1 \leqq A \leqq 1, \tag{10}$$

while an overbar indicates an image quantity with respect to the plane $z = 0$, *that is,* $\bar{\boldsymbol{U}} = \boldsymbol{U}(x, y, -z)$, *etc.* $\boldsymbol{n}$ *denotes a unit vector along the z-axis, while the superscripts* 1 *and* 2 *in round brackets refer respectively to the regions* $z > 0$ *and* $z < 0$. *Conditions* (7) *ensure the continuity of flow along* $z = 0$, *that is, continuous velocity and surface traction across the interface.*

Before proceeding to the proof, let us remark that the special results for the case of a semi-infinite fluid bounded by a rigid plane [6] can be obtained from the foregoing formulae by merely setting $A = 1$, while the case of a homogeneous infinite fluid is recoverable by setting $A = 0$.

Proof. Let (x', y', z') and (x'', y'', z'') be two other systems of rectangular Cartesian coordinates which are connected with the reference system (x, y, z) through the relations

$$\begin{aligned} x' &= x\cos\alpha + (z - h)\sin\alpha,\ \ y' = y,\ \ z' = -x\sin\alpha + (z - h)\cos\alpha,\\ x'' &= x\cos\alpha - (z + h)\sin\alpha,\ \ y'' = y,\ \ z'' = x\sin\alpha + (z + h)\cos\alpha, \end{aligned} \tag{11}$$

where α and h are constants. Thus the system (x', y', z') corresponds to a translation and rotation of axes, while the system (x'', y'', z'') is its image with respect to the plane $z = 0$. Equations (8–10) will be established by showing that they are satisfied by the fundamental solution for a uniform force applied on the plane $z' = 0$ over a finite plane area S containing the arbitrary point $x' = \xi$, $y' = \eta$.

Consider first the case when the force, $f_{x'}$, acts parallel to the x'-axis. Then, for the homogeneous infinite fluid, the velocity and pressure fields are given by [5, 9], after a straightforward coordinate transformation

$$U = \nabla(\Psi_0 + x\Psi_x + z\Psi_z) - 2(\Psi_x, 0, \Psi_z),$$
$$P = 2\mu^{(1)} f_1 \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial z}\right) I_1 \tag{12}$$

where

$$f_1 = -f_{x'}/8\pi\mu^{(1)},\ I_1 = \int_S \frac{1}{R_1}\, dS,\ J_1 = \int_S \frac{\xi}{R_1}\, dS,$$
$$R_1 = [(x'-\xi)^2 + (y'-\eta)^2 + z'^2]^{\frac{1}{2}}, \tag{13}$$

while

$$\Psi_0 = -f_1(J_1 + h\sin\alpha I_1),$$
$$(\Psi_x, \Psi_z) = f_1(\cos\alpha, \sin\alpha) I_1. \tag{14}$$

The double integrals I_1 and J_1 are the Newtonian potentials of surface distributions with densities $\rho(\boldsymbol{r}) = 1$, $\rho(\boldsymbol{r}) = \boldsymbol{r}$. For the calculation of the stress field, it is unnecessary to evaluate these integrals as they stand, since only their derivatives are required. However, the calculation of the velocity field requires an explicit knowledge of I_1. This presents no mathematical difficulty for some shapes of practical importance, for the double integrals can always be transformed into line integrals taken round the bounding curve of S by means of Stokes's integral theorem. Indeed, as shown by Sadowsky and Sternberg [20], among others [21], I_1 admits a representation in terms of complete and incomplete elliptic integrals if the loaded area S is a circle. On the other hand, if S is the rectangle

$$\{(\xi, \eta) \mid -a \leqq \xi \leqq a,\ -b \leqq \eta \leqq b\},$$

I_1 can be expressed in terms of logarithms and inverse tangents with the aid of an indefinite integral* of Kellogg [22]:

$$I_1 = (a - x')\ln\frac{R_{11} + b - y'}{R_{12} - b - y'} + (a + x')\ln\frac{R_{21} + b - y'}{R_{22} - b - y'}$$
$$+ (b - y')\ln\frac{R_{11} + a - x'}{R_{21} - a - x'} + (b + y')\ln\frac{R_{12} + a - x'}{R_{22} - a - x'}$$
$$- z'\left[\tan^{-1}\left(\frac{(a - x')(b - y')}{z'R_{11}}\right) + \tan^{-1}\left(\frac{(a - x')(b + y')}{z'R_{12}}\right)\right.$$
$$\left. + \tan^{-1}\left(\frac{(a + x')(b - y')}{z'R_{21}}\right) + \tan^{-1}\left(\frac{(a + x')(b + y')}{z'R_{22}}\right)\right] \tag{15}$$

* This indefinite integral contains an obvious misprint.

where

$$R_{ij}^2 = [x' + (-1)^i a]^2 + [y' + (-1)^j b]^2 + z'^2, \quad i, j = 1, 2. \tag{16}$$

To evaluate the derivatives of J_1 with the aid of Stokes' theorem, J_1 should be rewritten

$$J_1 = x' I_1 - \int_S \frac{\partial}{\partial \xi} R_1 \, dS. \tag{17}$$

The significance of the statement of our theorem is that once the necessary integrals for the homogeneous infinite fluid have been evaluated, the additional residual fields for the two-phase flow involve only the operation of *differentiation*, as will now be shown.

Thus let the regions $z > 0$ and $z < 0$ be occupied by immiscible fluids, and construct the required Papkovitch–Neuber potentials as follows:
For $z > 0$,

$$\begin{aligned} \Psi_0^{(1)} &= -f_1(J_1 + h \sin \alpha I_1) + f_1 A_1 (J_2 + h \sin \alpha I_2), \\ \Psi_x^{(1)} &= f_1 \cos \alpha (I_1 + A_2 I_2), \ \Psi_y^{(1)} = 0, \\ \Psi_z^{(1)} &= f_1 \sin \alpha (I_1 + A_3 I_2) - 2 f_1 A_3 \frac{\partial}{\partial z} (J_2 + h \sin \alpha I_2) \\ &\quad + 2 f_1 A_4 \cos \alpha \left(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right) I_2; \end{aligned} \tag{18}$$

and for $z < 0$,

$$\begin{aligned} \Psi_0^{(2)} &= -f_1 A_5 (J_1 + h \sin \alpha I_1), \\ \Psi_x^{(2)} &= f_1 A_5 \cos \alpha I_1, \ \Psi_y^{(2)} = 0, \ \Psi_z^{(2)} = f_1 A_5 \sin \alpha I_1, \end{aligned} \tag{19}$$

where

$$\begin{aligned} I_2 &= \int_S \frac{1}{R_2} \, dS, \ J_2 = \int_S \frac{\xi}{R_2} \, dS, \\ R_2 &= [(x'' - \xi)^2 + (y'' - \eta)^2 + z''^2]^{\frac{1}{2}}, \end{aligned} \tag{20}$$

the variables x'', y'' and z'' being given by equations (11). The constants A_i are to be determined from the satisfaction of the six interface conditions (7) which yields the peculiarly simple expressions

$$A_1 = A, \ A_2 = -A, \ A_3 = -A, \ A_4 = -A, \ A_5 = 1 - A, \tag{21}$$

where A is given by (10). By formally substituting equations (18) to (21) into (3) and then comparing the resulting expressions with equations (12) we conclude that the velocity and pressure fields in the two phases are connected with the basic fields in the homogeneous infinite fluid according to equations (8) and (9). A general conclusion which can also be drawn is the following:

Corollary. If Ψ_0 and $\boldsymbol{\Psi}$ denote the Papkovitch potentials due to an arbitrary steady motion of an incompressible homogeneous infinite viscous fluid such that all these functions have their singularities in the region $z > 0$, then on introducing a different viscous fluid into the region

$z < 0$ so that the conditions (7) are satisfied at the interface, the new Papkovitch potentials are given as follows:
For $z > 0$,

$$\Psi_0^{(1)} = \Psi_0 - A\bar{\Psi}_0,\ \Psi_x^{(1)} = \Psi_x - A\bar{\Psi}_x,\ \Psi_y^{(1)} = \Psi_y - A\bar{\Psi}_y,$$
$$\Psi_z^{(1)} = \Psi_z - A\bar{\Psi}_z - 2A\left[\frac{\partial}{\partial z}\bar{\Psi}_0 + \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)\bar{\Psi}_x + \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\bar{\Psi}_y\right]; \tag{22}$$

and for $z < 0$,

$$(\Psi_0^{(2)}, \boldsymbol{\Psi}^{(2)}) = (1 - A)(\Psi_0, \boldsymbol{\Psi}) \tag{23}$$

where A is once again given by (10).

It is also of interest to state that the stresses in the two phases can be written down without going through a great deal of mathematical analysis once the flow fields for the homogeneous infinite fluid are specified. Thus

$$\begin{aligned}
\sigma_{xx}^{(1)} &= \sigma_{xx} - A\bar{\sigma}_{xx} + 2A\left(z\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial z}\right)\bar{W},\\
\sigma_{yy}^{(1)} &= \sigma_{yy} - A\bar{\sigma}_{yy} + 2A\left(z\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial z}\right)\bar{W},\\
\sigma_{zz}^{(1)} &= \sigma_{zz} + A\bar{\sigma}_{zz} + 2Az\frac{\partial}{\partial z}\left(\frac{\partial \bar{W}}{\partial z} - \bar{P}\right),\\
\sigma_{zx}^{(1)} &= \sigma_{zx} + A\bar{\sigma}_{zx} + 2Az\frac{\partial^2}{\partial x\partial z}\bar{W},\\
\sigma_{zy}^{(1)} &= \sigma_{zy} + A\bar{\sigma}_{zy} + 2Az\frac{\partial^2}{\partial y\partial z}\bar{W},\\
\sigma_{xy}^{(1)} &= \sigma_{xy} - A\bar{\sigma}_{xy} + 2Az\frac{\partial^2}{\partial x\partial y}\bar{W};
\end{aligned} \tag{24}$$

$$\boldsymbol{\sigma}^{(2)} = (1 + A)\boldsymbol{\sigma}, \tag{25}$$

where

$$\bar{W} = 2\mu^{(1)}\bar{U}_z + z\bar{P}. \tag{26}$$

As in the velocity formulae (8–9), equations (24–26) show that the calculation of the desired stress components consists only of the superposition of the direct and differentiated images. It can be shown, but not easily, that this contrasts to the case of bonded isotropic semi-infinite solids where integral terms are encountered. If the motion of the homogeneous infinite fluid happens to be irrotational, as a consequence of the presence of sources or sinks, for example, equations (24–26) assume simple forms, since, in this special case, $P = \bar{P} = 0$.

An application can now be made to the case of a uniform force, $f_{y'}$, applied parallel to the y'-axis on the same area S in the interior of the fluid occupying $z > 0$. For the homogeneous infinite fluid, the non-vanishing Papkovitch potentials are

$$\Psi_0 = -f_2 K_1, \; \Psi_y = f_2 I_1, \; f_2 = -f_{y'}/8\pi\mu^{(1)}, \tag{27}$$

where I_1 is given by (13), while

$$K_1 = \int_S \frac{\eta}{R_1} dS = y' I_1 - \int_S \frac{\partial}{\partial \eta} R_1 dS. \tag{28}$$

Equations (22) and (23) therefore give the potentials for the two phases as follows:

$$\begin{aligned} &\Psi_0^{(1)} = -f_2(K_1 - AK_2), \; \Psi_x^{(1)} = 0, \; \Psi_y^{(1)} = f_2(I_1 - AI_2), \\ &\Psi_z^{(1)} = 2f_2 A\left[\frac{\partial}{\partial z} K_2 - \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) I_2\right)\right]; \end{aligned} \tag{29}$$

$$\begin{aligned} &\Psi_0^{(2)} = -f_2(1 - A)K_1, \; \Psi_x^{(2)} = 0, \\ &\Psi_y^{(2)} = f_2(1 - A) I_1, \; \Psi_z^{(2)} = 0. \end{aligned} \tag{30}$$

where

$$K_2 = \int_S \frac{\eta}{R_2} \equiv y'' I_2 - \int_S \frac{\partial}{\partial \eta} R_2 dS. \tag{31}$$

When a uniform force, $f_{z'}$, is applied on the same area, parallel to the z'-axis, we have the following results

$$\begin{aligned} &\Psi_0^{(1)} = -f_3 h \cos\alpha(I_1 - AI_2), \; \Psi_x^{(1)} = -f_3 \sin\alpha(I_1 - AI_2), \; \Psi_y^{(1)} = 0, \\ &\Psi_z^{(1)} = f_3 \cos\alpha(I_1 - AI_2) + 2f_3 A\left[h\cos\alpha\frac{\partial}{\partial z} + \sin\alpha\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)\right] I_2, \\ &f_3 = -f_{z'}/8\pi\mu^{(1)}; \end{aligned} \tag{32}$$

$$\begin{aligned} &\Psi_0^{(2)} = -f_3 h(1 - A)\cos\alpha I_1, \; \Psi_x^{(2)} = -f_3(1 - A)\sin\alpha I_1, \\ &\Psi_z^{(2)} = f_3(1 - A)\cos\alpha I_1, \; \Psi_y^{(2)} = 0, \end{aligned} \tag{33}$$

where I_1 and I_2 are, as previously defined, the Newtonian potentials of surface distributions, their properties being well-known [20–21].

4. Concluding remarks

A simple method for the calculation of singularity flow fields in two immiscible semi-infinite viscous fluids has been developed. Our main objective has been to lay the mathematical foundations for an efficient treatment of the problem of a transforming inclusion in a stratified half-space consisting of a fluid overlying another fluid, the top surface of the overlying fluid being either stress- or velocity-free. Explicit expressions based upon a repeated application of the theorem offered here will be supplied in a separate paper. The solution of the problem of a spherical liquid drop perturbing a *general* Stokes' flow has already been submitted for publication elsewhere.

Let us finally remark that the theorem presented here remains in force for two-dimensional investigations dealing with immiscible fluids occupying the domains $z > 0$ and $z < 0$, where x and z now stand for two-dimensional rectangular Cartesian coordinates. We merely need to ignore U_y in equations (8–9), while Ψ_y must also be set equal to zero in the formulae (22–23). An application can then be made to the fundamental case of a point force $\boldsymbol{F} = (F_x, F_z)$ applied at the point $x = \xi$, $x = \eta$ in the interior of the fluid occupying $z > 0$. For the homogeneous infinite fluid, the basic potentials required in (22–23) are

$$(\Psi_0, \Psi_x, \Psi_z) = \varepsilon(\eta F_z, -F_x, -F_z) \log r_1 \tag{34}$$

where ε is a parametric constant, while $r_1 = [(x - \xi)^2 + (z - \eta)^2]^{\frac{1}{2}}$. For given sources and vortices, $\Psi_x = \Psi_z \equiv 0$, while Ψ_0 is a scalar multiple of $\log r_1$ and

$$\arctan [(z - \eta) | (x - \xi)],$$

respectively.

Furthermore, an interesting analogy exists in this two-dimensional case if we proceed to the limit $A \to 1$, corresponding to a semi-infinite fluid $z > 0$ bounded by the rigid plane $z = 0$ and containing singularities in its interior. According to formulae (8), such a situation yields

$$U_z^{(1)} = U_z - \left[1 - 2z \frac{\partial}{\partial z} + z^2 \nabla^2\right] \bar{U}_z. \tag{35}$$

On the other hand, when an isotropic homogeneous semi-infinite plate $z > 0$ is rigidly clamped along $z = 0$ and is subjected to any concentrated transverse load in its interior, the induced flexural deflection $\omega^{(1)}$ is connected with the corresponding deflection ω in the homogeneous infinite plate through

$$\omega^{(1)} = \omega - \left[1 - 2z \frac{\partial}{\partial z} + z^2 \nabla^2\right] \bar{\omega} \tag{36}$$

But we can also show that the extensional Airy stress function $\phi^{(1)}$ due to any body force distribution, dislocation or thermal dilations in the interior of a semi-infinite plate with a stress-free edge satisfies

$$\phi^{(1)} = \phi - \left[1 - 2z \frac{\partial}{\partial z} + z^2 \nabla^2\right] \bar{\phi}. \tag{37}$$

The analogy between equations (36–37) is complete. Although equations (36–37) are merely *presented* here, their derivation is simple, following the procedure contained in [15].

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